77. Proposed by J. OWEN MAHONEY, B. E., M. Sc., Professor of Mathematics and Science, Cooper Training School, Carthage, Tex.; and ELMER SCHUYLER, Reading, Pa.

A and B are two inaccurate mathematicians whose chance of solving a given question correctly is 1/8 and 1/12 respectively; if they obtain the same result, and if it is 1000 to 1 against their making the same mistake, find the chance that the result is correct. [From *Hall and Knight's Algebra.*]

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; COOPER D. SCHMITT, A. M., Ph. D., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; and L. C. WALKER, Instructor in Mathematics, Leland Stanford, Jr., University, Palo Alto, Cal.

The chance that A and B both get the correct result is $\frac{1}{8}\times\frac{1}{12}=\frac{1}{96}$.

The chance that both get the wrong result is $\frac{7}{8}\times\frac{11}{12}=\frac{77}{96}$.

The chance that they both get the *same* wrong result is $\frac{1}{1001}\times\frac{77}{96}=\frac{1}{13.96}$ $=\frac{1}{1248}$.

$\therefore$ The chance that the result is correct : the chance that the result is not correct :: 13 : 1.

$\therefore$ The required chance is $\frac{13}{14}$.

Also the required chance is $(\frac{1}{96})/(\frac{1}{96}+\frac{1}{1248})=\frac{13}{14}$.

78. Proposed by CHAS. E. MYERS, Canton, O.

Two witnesses, A and B, both make the statement that an event happened in a particular way (two ways being possible). Find the probability of the truth of the statement.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let p, q be the chances that A, B speak the truth, respectively.

Then the chance of the truth of the statement is

$$\frac{pq}{pq+(1-p)(1-q)}=c.$$

Now $p=q=\frac{1}{2}$. $\therefore c=\frac{1}{4}/[\frac{1}{4}+(1-\frac{1}{2})(1-\frac{1}{2})]=\frac{1}{4}/\frac{1}{2}=\frac{1}{2}$.

MISCELLANEOUS.

72. Proposed by E. D. ROE, Jr., A. M., Ph. D., Professor of Mathematics, Oberlin College, Oberlin, Ohio.

If a, b, and c are integers, and

$$\left.\begin{Bmatrix} b,\ c-b,\ c-1 \\ c-a-1 \\ c-a-1 \end{Bmatrix}\right.>0,$$

prove that the sum of the series,

$$1+\frac{a.b}{1.c}+\frac{a(a+b).b(b+1)}{1.2.c(c+1)}+\frac{a(a+1)(a+2).b(b+1)(b+2)}{1.2.3c(c+1)(c+2)}+\ldots\ldots$$

is equal to

$$\frac{(c-1)!\,(c-a-b-1)!}{(c-a-1)!\,(c-b-1)!}.$$

I. **Solution by the PROPOSER.**

Let $fx=\int_0^1 u^{b-1}(1-u)^{c-b-1}(1-xu)^{-a}du.$

If $0<x<1$, $0\leqq u\leqq 1$, $(1-xu)^{-a}=1+\frac{a}{1}xu+\frac{a(a+1)}{1.2}x^2u^2$

$$+\frac{a(a+1)(a+2)}{1.2.3}x^3u^3+\ldots\ldots$$

and therefore

$$fx=\int_0^1 u^{b-1}(1-u)^{c-b-1}du+\frac{a}{1}x\int_0^1 u^b(1-u)^{c-b-1}du$$

$$+\frac{a(a+1)}{1.2}x^2\int_0^1 u^{b+1}(1-u)^{c-b-1}du+\ldots\ldots$$

$$=B(b,\ c-b)+\frac{a}{1}xB(b+1,\ c-b)+\frac{a(a+1)}{1.2}x^2B(b+2,\ c-b)+\ldots\ldots(1),$$

where $B(m,n)=\int_0^1 u^{m-1}(1-u)^{n-1}du$, is known as the Beta Function or First Eulerian Integral. (Cf. Byerly's *Integral Calculus*, page 109.)

Now we have l. c. page 110,

$$B(m,\ n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},\ \text{with}\ \Gamma(n+1)=n\Gamma(n),\ \text{where}\ \Gamma(n)=\int_0^\infty u^{n-1}e^{-u}du,$$

and is known as the Gamma Function or Second Eulerian Integral.

We have therefore,

$$B(m+1,\ n)=\frac{m\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)}=\frac{m}{m+n}B(m,n).$$

If in this formula $m=b$, $n=c-b$, we get $B(b+1,\ c-b)=(b/c)B(b,\ c-b)$.
Similarly,

$$B(b+2,\ c-b)=\frac{b+1}{c+1}B(b+1,\ c-b)=\frac{b(b+1)}{c(c+1)}B(b,\ c-b).$$

$$B(b+3,\ c-b)=\frac{b+2}{c+2}B(b+2,\ c-b)=\frac{b(b+1)(b+2)}{c(c+1)(c+1)}B(b,\ c-b),$$

and by mathematical induction,

$$B(b+n,\ c-b)=\frac{b(b+1)\ldots\ldots(b+n)}{c(c+1)\ldots\ldots(c+n)}B(b,\ c-b).$$

By substituting these values in (1), we get

$$\int_0^1 u^{b-1}(1-u)^{c-b-1}(1-xu)^{-a}du=B(b,\ c-b)\Big(1+\frac{a.b}{1.c}x+\frac{a(a+1).b(b+1)}{1.2\ \ c(c+1)}x^2$$

$$+\frac{a(a+1)(a+2).b(b+1)(b+2)}{1.2.3\ \ c(c+1)(c+2)}x^3+\ldots\ldots\Big)\ldots\ldots(2).$$

Taking the limit of both members of (2) as $x=1$, we have

$$\Big[\int_0^1 u^{b-1}(1-u)^{c-a-b-1}du=\Big]B(b,\ c-a-b)=B(b,\ c-b)\Big[1+\frac{a.b}{1.c}$$

$$+\frac{a(a+1).\ b(b+1)}{1.2.c(c+1)}+\frac{a(a+1)(a+2).b(b+1)(b+2)}{1.2.3\ \ c(c+1)(c+2)}+\ \ldots\ldots\Big]\ldots\ldots(3),$$

or $$\Big[1+\frac{a.b}{1.c}+\frac{a(a+1).b(b+1)}{1.2\ \ c(c+1)}+\frac{a(a+1)(a+2).b(b+1)(b+2)}{1.2.3\ \ c(c+1)(c+2)}+\ldots\ldots\Big]$$

$$=\frac{B(b,\ c-a-b)}{B(b,\ c-b)}=\frac{\Gamma(b)\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(b)\Gamma(c-b)}=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

$$=\frac{(c-1)!\,(c-a-b-1)!}{(c-a-1)!\,(c-b-1)!}\ldots\ldots(4),$$

if a, b, and c are integers, and the inequalities stated in the problem are satisfied.

II Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $$y_{cx}=1+\frac{ab}{1.c}x+\frac{a[a+1]b[b+1]}{1.2.c[c+1]}x^2+\frac{a[a+1][a+2]b[b+1][b+2]}{1.2.3.c[c+1][c+1]}x^3+\ldots.$$

$$\therefore\ \frac{dy_{cx}}{dx}=\frac{ab}{1.c}+\frac{a[a+1]b[b+1]}{1.c[c+1]}x+\frac{a[a+1][a+2]b[b+1][b+2]}{1.2.c[c+1][c+2]}x^2+\ldots.$$

$$=\frac{ab}{c}\Big[1+\frac{[a+1][b+]}{1.[c+1]}x+\frac{[a+1][a+2]\,b[b+1][b+2]}{1.2.c[c+1][c+2]}x^2+\ \ldots.\Big]$$

$$\frac{d^2y_{cx}}{dx^2}=\frac{a[a+1]b[b+1]}{c.[c+1]}\Big[1+\frac{[a+2][b+2]}{2.[c+2]}x+\ \ldots.\Big]$$

$$\therefore\ x(1-x)\frac{d^2y_{cy}}{dx^2}+\{c-[a+b+1]x\}\frac{dy_{cx}}{dx}=aby_{cx}\ldots\ldots(1).$$

$$y_{(c-1)x}=1+\frac{ab}{1.[c-1]}x+\frac{a[a+1]b[b+1]}{1.2.[c-1]c}x^2+\frac{a[a+1][a+2]\,b[b+1][b+2]}{1.2.3.[c-1]c[c+1]}x^2+\ldots$$

$$y_{cx}-y_{(c-1)x}=-\frac{abx}{c[c-1]}\left[1+\frac{[a+1][b+1]}{1.[c+1]}x+\frac{[a+1][a+2][b+1][b+2]}{1.2.[c+1][c+2]}x^2+.\right]$$

$$\therefore\ y_{(c-1)x}-y_{cx}=\frac{abx}{c[c-1]}\cdot\frac{c}{ab}\frac{dy_{cx}}{dx}=\frac{x}{c-1}\frac{dy_{cx}}{dx}\ldots\ldots(2).$$

If we make x=unity in (1) we get

$$\frac{dy_c}{dx}=\frac{aby_c}{c-a-b-1}.\qquad\therefore\ y_{c-1}-y_c=\frac{aby_c}{[c-1][c-a-b-1]}.$$

$$\therefore\ y_{c-1}=\frac{[c-1][c-a-b-1]+ab}{[c-1][c-a-b-1]}y_c=\frac{[c-a-1][c-b-1]}{[c-1][c-a-b-1)}y_c.$$

By symmetry,

$$y_c=\frac{[c-a][c-b]}{c[c-a-b]}y_{c+1},\quad y_{c+1}=\frac{[c-a+1][c-b+1]}{[c+1][c-a-b+1]}y_{c+2}\ldots\ldots(3,\ 4).$$

(4) in (3) gives

$$y_c=\frac{[c-a][c-a+1][c-b][c-b+1]}{c[c+1][c-a-b][c-a-b+1]}y_{c+2}.$$

$$\therefore\ y_c=\frac{[c-a][c-a+1]\ldots[c-a-1+n][c-b][c-b+1]\ldots[c-b-1+n]}{c[c+1]\ldots[c-1+n][c-a-b][c-a-b+1]\ldots[c-a-b-1+n]}y_{c+n}.$$

Let $[c-a-1]=s$, $[c-b-1]=t$, $[c-1]=u$, $[c-a-b-1]=v$. Then

$$_c=\frac{\dfrac{[s+1][s+2]\ldots\ldots[s+n]}{1.2.3\ \ldots\ n}\cdot\dfrac{[t+1][t+2]\ldots\ldots[t+n]}{1.2.3\ldots\ldots n}}{\dfrac{[u+1][u+2]\ldots\ldots[u+n]}{1.2.3\ldots\ldots n}\cdot\dfrac{[v+1][v+2]\ldots\ldots[v+n]}{1.2.3\ \ldots\ldots n}}y_{c+n}$$

$$=\frac{\frac{[1+(1/n)][1+(2/n)]\,..\,[1+(s/n)]}{1.2.3......s}\cdot\frac{[1+(1/n)][1+(2/n)]...[1+(t/n)]}{1.2.3......t}}{\frac{[1+(1/n)][1+(2/n)]...[1+(u/n)]}{1.2.3......u}\cdot\frac{[1+(1/n)][1+(2/n)]\,..\,[1+(v/n)]}{1.2.3....\,.v}}y_{c+n}.$$

Let $n=\infty$, then $y_{c+n}=1+\frac{ab}{c+\infty}+\frac{a[a+1]b[b+1)}{1.2[c+\infty][c+\infty+1]}+......$

$\therefore y_{c+n}=1.$

$\therefore y_c=\frac{u!\,v!}{s!\,t!}=\frac{[c-1]!\;[c-a-b-1]!}{[c-a-1]!\;[c-b-1]!}.$

But $y_c=1+\frac{ab}{1.c}+\frac{a[a+1]b[b+1]}{1.2.c[c+1]}+......$

Therefore, etc. [See *Forsyth's Differential Equations*, chápter VI, page 185, for treatment of this series.]

PROBLEMS FOR SOLUTION.

ARITHMETIC.

112. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Suppose 10% traction stock is 20% better in the market than 5% mining stock; if my income be $500 from each, how much money have I paid for each, the whole investment bringing $6\frac{2}{3}$%?

123. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

If $m=2$ cents be the interest on $M=100$ cents for $p=40$ days, find the yearly rate per cent.

*** Solutions of these problems should be sent to B. F. Finkel not later than January 10.

ALGEBRA.

111. Proposed by ARTEMAS MARTIN, A. M., Ph. D., LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Solve the equation $x(y+z)=a(x+y+z)$, $y(x+z)=b(x+y+z)$, $z(x+y)=c(x+y+z)$.